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New Operator Identities and Integration Formulas Regarding to Hermite Polynomials Obtained via the Operator Ordering Method

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Abstract Based on the technique of integration within an ordered product (IWOP) of operators we show that the operator ordering method can lead us to derive new operator identities and new integration formulas regarding to Hermite polynomials.

Keywords Hermite polynomials · IWOP technique · New Integration Formulas

1 Introduction

In quantum mechanics most operators do not commute, so arranging operators into their ordered product form (normal ordering, anti-normal ordering, Weyl ordering) often bothers theoretician physicists. On the other hand, usual Newton-Leibniz integration rule only applies to commuting c-functions, integration over operators made of Dirac ket-bra need new technique. This is our newly developed technique of integration within an ordered product (IWOP) of operators which is summarized in two recent Review Articles [1, 2]. The technique is proposed via the route of fashioning Dirac's symbolic method [3] and representation theory. The terminology "symbolic method" was first shown in the preface of Dirac's book 'The Principle of Quantum Mechanics'. Dirac wrote: "... The symbolic method, which deals directly in an abstract way with the quantities of fundamental importance..., however, seems to go more deeply into the nature of things. It enables one to express the physical law in a neat and concise way, and will probably be increasingly used in the future as it becomes better understood and its own special mathematics gets developed." The IWOP technique not only help people to better understand the symbolic method, but also directly

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develops its special mathematics, i.e. the Newton-Leibniz integration rule can now be directly applied to the operators in Dirac's ket-bra form. As a result, many operator identities and some new quantum mechanical representations can be derived. In this work we shall demonstrate some applications of the fundamental operator identity $H_n(X) = 2^n : X^n :$ [1], where $X = (a + a^\dagger)/\sqrt{2}$ is the coordinate operator, $[a, a^\dagger] = 1$, $H_n(X)$ is the Hermite polynomial operator, :: denotes the normal ordering. We shall show that the IWOP technique can not only simplify the derivation of the properties of Hermite polynomials, but also can directly lead to some new operator identities and new integration formulas regarding to $H_n(x)$. We name this approach the operator ordering method. The work is arranged as follows. In Sect. 2 we briefly review how the main properties of $H_n(x)$ can be easily derived by simply using $H_n(X) = 2^n : X^n :$. In Sects. 3 and 5 we show how some integration formulas can be derived by operator ordering method. In Sects. 4 and 6 we derive some new operator identities. Since Hermite polynomials are closely related to other polynomials, such as Laguerre functions, Legendre polynomials and Bessel functions, so this paper's content may have potential uses for studying them also by virtue of the operator ordering method.

2 Main Properties of $H_n(x)$ Derived by $H_n(X) = 2^n : X^n :$

The series definition of Hermite polynomials is

$$H_n(x) = 2^n \sum_{k=0}^{[n/2]} \frac{(-1)^k n!}{2^{2k} k!(n-2k)!} x^{n-2k}. \tag{1}$$

Using its generating function we can normally ordered expand

$$e^{2\lambda X - \lambda^2} = \sum_{n=0} \frac{\lambda^n}{n!} H_n(X) =: e^{2\lambda X} :=: \sum_{n=0} \frac{(2X\lambda)^n}{n!} : . \tag{2}$$

Comparing the power λ^n of the two sides of (1) and (2) we obtain an easily remembered operator formula

$$H_n(X) = 2^n : X^n : . \tag{3}$$

It is obvious that within :: we have

$$n : X^{n-1} :=: \frac{d}{dX} X^n := n : X^{n-1} : . \tag{4}$$

On the other hand, from the well-known relation $\frac{d}{dX} H_n(X) = 2n H_{n-1}(X)$, we know

$$2^n \frac{d}{dX} : X^n := \frac{d}{dX} H_n(X) = \frac{d}{dX} : 2^n X^n := 2n H_{n-1}(X) = n 2^n : X^{n-1} : . \tag{5}$$

Comparing (4) and (5) implies

$$\frac{d}{dX} : X^n :=: \frac{d}{dX} X^n : . \tag{6}$$

This property is very important regarding to the normal ordering. Using (6) and (3) we can easily derive main properties of $H_n(X)$ (so can for $H_n(x)$) by simple bosonic operator algebraic method. For instance

$$\left(\frac{d}{dX}\right)^s H_n(X) = 2^n \left(\frac{d}{dX}\right)^s : X^n := 2^n : \left(\frac{d}{dX}\right)^s X^n :$$
$$= \frac{2^n n!}{(n-s)!} : X^{n-s} := \frac{2^s n!}{(n-s)!} H_{n-s}(X). \tag{7}$$

From

$$[: f(a,a^\dagger) :, a] = - : \frac{\partial}{\partial a^\dagger} f(a,a^\dagger) :, \qquad [: f(a,a^\dagger) :, a^\dagger] =: \frac{\partial}{\partial a} f(a,a^\dagger) :, \tag{8}$$

we can calculate the commutative relation,

$$[: X^n :, a] = - : \frac{\partial}{\partial a^\dagger} X^n := -\frac{1}{\sqrt{2}} n : X^{n-1} :, \tag{9}$$

$$[: X^n :, a^\dagger] = : \frac{\partial}{\partial a} X^n := \frac{1}{\sqrt{2}} n : X^{n-1} : . \tag{10}$$

It then follows

$$: X^n : = \frac{1}{\sqrt{2}} (a^\dagger : X^{n-1} : + : X^{n-1} : a)$$
$$= \frac{1}{\sqrt{2}} \left[a^\dagger : X^{n-1} : + a : X^{n-1} : - \frac{1}{\sqrt{2}} (n-1) : X^{n-2} : \right]$$
$$= X : X^{n-1} : - \frac{1}{2} (n-1) : X^{n-2} : . \tag{11}$$

Equation (11) implies the recursive relation

$$H_n(X) = 2X H_{n-1}(X) - 2(n-1) H_{n-2}(X). \tag{12}$$

From (6) we also have

$$\frac{d^2}{dX^2} : X^n := n(n-1) : X^{n-2} := 2n(X : X^{n-1} : - : X^n :), \tag{13}$$

comparing (3) and (13) we see the Hermite equation

$$H_n''(x) - 2X H_n'(x) + 2n H(x) = 0. \tag{14}$$

3 Integration Formulas Derived by the Operator Ordering Method

Using the completeness relation of coordinate eigenstate $|x\rangle$,

$$\int_{-\infty}^{\infty} dx |x\rangle\langle x| = \frac{1}{\sqrt{\pi}} \int_{-\infty}^{\infty} dx : e^{-(x-X)^2} := 1, \tag{15}$$

where

$$|x\rangle = \pi^{-1/4} \exp\left[-\frac{x^2}{2} + \sqrt{2}xa^\dagger - \frac{a^{\dagger 2}}{2}\right]|0\rangle, \tag{16}$$

and we have used the normal ordering of vaccum projector $|0\rangle\langle 0| =: e^{-a^\dagger a} :$. So combining (15) and (3) we have

$$H_n(X) = \int_{-\infty}^{\infty} dx|x\rangle\langle x|H_n(x) = \frac{1}{\sqrt{\pi}} \int_{-\infty}^{\infty} dx : e^{-(x-X)^2} : H_n(x) =: 2^n X^n :, \tag{17}$$

which implies the integration formula

$$\frac{1}{\sqrt{\pi}} \int_{-\infty}^{\infty} dx e^{-(x-y)^2} H_n(x) = 2^n y^n, \tag{18}$$

this result is in agreement with [4]. Further, from

$$e^{-\lambda X} =: e^{-\lambda X} : e^{\lambda^2/4} =: e^{\lambda^2/4 - \lambda X} := \sum_{n=0}^{\infty} \frac{(i\lambda/2)^n}{n!} : H_n(iX) :, \tag{19}$$

and (3) we can also have the expansion in terms of Hermite polynomials

$$e^{-\lambda X} = e^{\lambda^2/4} : e^{-\lambda X} := e^{\lambda^2/4} \sum_{n=0}^{\infty} : \frac{(-\lambda X)^n}{n!} := e^{\lambda^2/4} \sum_{n=0}^{\infty} \frac{(-\lambda)^n}{2^n n!} H_n(X). \tag{20}$$

On the other hand

$$e^{-\lambda X} = \sum_{n=0}^{\infty} \frac{(-\lambda X)^n}{n!}. \tag{21}$$

Comparing the two sides of (20) and (21) we have another operator identity

$$X^n = (2i)^{-n} : H_n(iX) : . \tag{22}$$

Note using (15), (22) and (3) we have

$$\begin{aligned} X^n &= \int_{-\infty}^{\infty} dx x^n |x\rangle\langle x| = \frac{1}{\sqrt{\pi}} \int_{-\infty}^{\infty} dx : e^{-(x-X)^2} x^n := (2i)^{-n} : H_n(iX) : \\ &= \sum_{k=0}^{[n/2]} \frac{n!}{2^{2k} k!(n-2k)!} : X^{n-2k} := \sum_{k=0}^{[n/2]} \frac{n!}{2^n k!(n-2k)!} H_{n-2k}(X), \end{aligned} \tag{23}$$

from which we can infer the integration formula

$$\frac{1}{\sqrt{\pi}} \int_{-\infty}^{\infty} dx e^{-(x-y)^2} x^n = (2i)^{-n} H_n(iy) = \sum_{k=0}^{[n/2]} \frac{n!}{2^{2k} k!(n-2k)!} y^{n-2k}. \tag{24}$$

In Sect. 4 we shall consider more complicated integration formulas regarding to $H_n(x)$.

4 Product Formula of Two Normally Ordered Operators

We now study what is the normally ordered form of product of two normally ordered operators $:X^p:$ and $:X^q:$? This can be examined as

$$\begin{aligned} :X^p::X^q: &= 2^{-(p+q)/2} :(a+a^\dagger)^p::(a+a^\dagger)^q: \\ &= 2^{-(p+q)/2}\sum_l\sum_k \frac{p!}{(p-l)!l!}\frac{q!}{(q-k)!k!}a^{\dagger p-l}a^l a^{\dagger k}a^{q-k}. \end{aligned} \tag{25}$$

Using the normally ordered form of the completeness relation of coherent state [5–7]

$$\int\frac{d^2z}{\pi}|z\rangle\langle z| = \int\frac{d^2z}{\pi}:\exp[-|z|^2+za^\dagger+z^*a-a^\dagger a]:=1 \tag{26}$$

we can easily derive normal ordering of $a^l a^{\dagger k}$,

$$\begin{aligned} a^l a^{\dagger k} &= \int\frac{d^2z}{\pi}z^l|z\rangle\langle z|z^{*k} = \int\frac{d^2z}{\pi}z^l z^{*k}:\exp[-|z|^2+za^\dagger+z^*a-a^\dagger a]: \\ &= \sum_{s=0}^{\min[l,k]}\frac{l!k!a^{\dagger k-s}a^{l-s}}{s!(l-s)!(k-s)!}. \end{aligned} \tag{27}$$

Substituting (27) into (25) we have

$$\begin{aligned} &:X^p::X^q: \\ &= 2^{-(p+q)/2}\sum_l\sum_k\sum_{s=0}^{\min[l,k]}\frac{p!}{(p-l)!l!}a^{\dagger p-l}\frac{l!k!a^{\dagger k-s}a^{l-s}}{s!(l-s)!(k-s)!}\frac{q!}{(q-k)!k!}a^{q-k} \\ &= 2^{-(p+q)/2}p!q!\sum_{s=0}\sum_l\sum_k\frac{1}{(p-l)!(l-s)!s!(q-k)!(k-s)!}a^{\dagger p-l+k-s}a^{q-k+l-s} \\ &= 2^{-(p+q)/2}p!q!\sum_{s=0}\sum_l\sum_k\binom{p-s}{l-s}\binom{q-s}{k-s}a^{\dagger p-l+k-s}a^{q-k+l-s}\frac{1}{s!(p-s)!(q-s)!} \\ &= p!q!\sum_{s=0}\frac{:X^{p+q-2s}:}{2^s(p-s)!s!(q-s)!}. \end{aligned} \tag{28}$$

It then follows

$$\langle z|:X^p:\int\frac{d^2z'}{\pi}|z'\rangle\langle z'|:X^q:|z\rangle = p!q!\sum_{s=0}\langle z|\frac{:X^{p+q-2s}:}{2^s(p-s)!s!(q-s)!}|z\rangle, \tag{29}$$

so we have another integration formula

$$\begin{aligned} &\int\frac{d^2z'}{\pi}\exp\{-|z|^2-|z'|^2+zz'^*+z^*z'\}(z'+z^*)^p(z+z'^*)^q \\ &\quad = p!q!\sum_{s=0}\frac{1}{(p-s)!s!(q-s)!}(z+z^*)^{p+q-2s}. \end{aligned} \tag{30}$$

This is the normally ordered form of $:X^p::X^q:$, which is a new operator identity. We also obtain the expansion of $e^{-\lambda X^2}$ in terms of $H_{2n}(X)$

$$\begin{aligned} e^{-\lambda X^2} &= \int_{-\infty}^{\infty} dx|x\rangle\langle x|e^{-\lambda x^2} = \frac{1}{\sqrt{\pi}}\int_{-\infty}^{\infty} dx :e^{-(x-X)^2-\lambda x^2}: \\ &= \frac{1}{\sqrt{1+\lambda}} :\exp\left[\frac{-\lambda}{1+\lambda}X^2\right]: \\ &= \frac{1}{\sqrt{1+\lambda}}\sum_{n=0}^{\infty}\frac{1}{2^{2n}n!}\left(\frac{-\lambda}{1+\lambda}\right)^n H_{2n}(X). \end{aligned} \tag{31}$$

5 Some Complicated Integration Formulas Derived by Operator Ordering Method

Equations (28) and (3) indicates

$$H_p(X)H_q(X) = p!q!\sum_{s=0}\frac{2^s H_{p+q-2s}(X)}{(p-s)!s!(q-s)!}, \tag{32}$$

which can be checked with mathematical induction [8]. On the other hand, using (15) we have

$$\begin{aligned} 2^{p+q}:X^p::X^q: = H_p(X)H_q(X) &= \int_{-\infty}^{\infty} dx|x\rangle\langle x|H_p(x)H_q(x) \\ &= \frac{1}{\sqrt{\pi}}\int_{-\infty}^{\infty} dx :e^{-(x-X)^2}: H_p(x)H_q(x) \\ &= p!q!\sum_{s=0}\frac{2^{p+q}:X^{p+q-2s}:}{2^s(p-s)!s!(q-s)!}, \end{aligned} \tag{33}$$

from which we obtain the integration formula

$$\frac{1}{\sqrt{\pi}}\int_{-\infty}^{\infty} dx e^{-(x-y)^2}H_p(x)H_q(x) = 2^{p+q}p!q!\sum_{s=0}\frac{y^{p+q-2s}}{2^s(p-s)!s!(q-s)!}.$$

In particular, when $q=0$, $H_0(x)=1$, (33) reduces to (17); when $y=0$, in (33) only the term of $2s=p+q$ survives, which in turn leads to $(p-s)!=(\frac{p-q}{2})!$, $(q-s)!=(\frac{q-p}{2})!$, since they are both in the denominator, so $q=p$, we see

$$\frac{1}{\sqrt{\pi}}\int_{-\infty}^{\infty} dx e^{-x^2}H_p(x)H_q(x) = \delta_{p,q}q!2^q, \tag{34}$$

as expected. Further, from (32) and (33) we have

$$\begin{aligned} &\frac{1}{\sqrt{\pi}}\int_{-\infty}^{\infty} dx e^{-(x-y)^2}H_p(x)H_q(x)H_r(x) \\ &= \frac{1}{\sqrt{\pi}}p!q!\sum_{n=0}\frac{2^n}{(p-n)!n!(q-n)!}\int_{-\infty}^{\infty} dx e^{-(x-y)^2}H_{p+q-2n}(x)H_r(x) \end{aligned}$$

$$= p!q!r!\sum_{n=0}\frac{(p+q-2n)!}{(p-n)!n!(q-n)!}\sum_{s=0}\frac{2^{p+q+r-n-s}y^{p+q+r-2n-2s}}{(p+q-2n-s)!s!(r-s)!}. \quad (35)$$

Especially when $y=0$, we have $p+q+r=2n+2s$, and $n=(p+q-r)/2$, so

$$\frac{1}{\sqrt{\pi}}\int_{-\infty}^{\infty}H_p(x)H_q(x)H_r(x)e^{-x^2}dx = \frac{2^{(p+q+r)/2}p!q!r!}{(\frac{p+q-r}{2})!(\frac{q+r-p}{2})!(\frac{p+r-q}{2})!}, \quad (36)$$

as expected. Note $p+q+r$ is even and the sum of any two of p,q,r is not smaller that the third. Combining (15), (23), (3), (37) and (34) we also derive the new operator ordering formula

$$\begin{aligned} X^n H_p(X) &= \frac{1}{\sqrt{\pi}}\int_{-\infty}^{\infty}dx x^n H_p(x) :e^{-(x-X)^2}: \\ &= \frac{1}{\sqrt{\pi}}\sum_{k=0}^{[n/2]}\frac{n!}{2^n k!(n-2k)!}\int_{-\infty}^{\infty}dx H_{n-2k}(x)H_p(x) :e^{-(x-X)^2}: \\ &= \frac{1}{\sqrt{\pi}}\sum_{k=0}^{[n/2]}\frac{n!}{2^n k!}p!\sum_{s=0}\frac{:X^{n-2k+p-2s}:}{2^s(p-s)!s!(n-2k-s)!}. \end{aligned} \quad (37)$$

6 Hermite Polynomial Expansion of $\frac{1}{X}$

In [8] we have derived the normally ordered expansion of $\frac{1}{X}$,

$$\frac{1}{X} = \sqrt{\pi}\sum\frac{(-1)^n}{\Gamma(n+\frac{3}{2})} :X^{2n+1}: \quad (38)$$

where

$$\Gamma\left(n+\frac{3}{2}\right) = \sqrt{\pi}2^{-(n+1)}(2n+1)!!. \quad (39)$$

Using (3) we have

$$\frac{1}{X} = \sum_{n=0}\frac{(-1)^n}{2^n(2n+1)!!}H_{2n+1}(X). \quad (40)$$

This can be checked by using

$$XH_{2n+1} = \frac{1}{2}(H_{2n+2}+H'_{2n+1}) = \frac{1}{2}[H_{2n+2}+2(2n+1)H_{2n}], \quad (41)$$

so

$$\begin{aligned} X\frac{1}{X} &= \frac{1}{2}\sum_{n=0}\frac{(-1)^n}{2^n(2n+1)!!}[H_{2n+2}+2(2n+1)H_{2n}] \\ &= \sum_{n=0}\frac{(-1)^n}{2^{n+1}(2n+1)!!}H_{2n+2}+\sum_{n=1}\frac{(-1)^n}{2^n(2n-1)!!}H_{2n}+1 \\ &= \sum_{n=0}\frac{(-1)^n}{2^{n+1}(2n+1)!!}H_{2n+2}+\sum_{n=0}\frac{(-1)^{n+1}}{2^{n+1}(2n+1)!!}H_{2n+2}+1 = 1. \end{aligned} \quad (42)$$

Further using (6) and (41) we have

$$\frac{1}{X^2} = -\frac{d}{dX}\frac{1}{X} = \sqrt{\pi}\sum_{n=0}\frac{(-1)^{n+1}(2n+1)}{\Gamma(n+\frac{3}{2})}H_{2n}(X), \tag{43}$$

so

$$\begin{aligned}&\sum_{n=0}\sum_{n'=0}\frac{(-1)^{n+n'}}{2^{n+n'}(2n+1)!!(2n'+1)!!}H_{2n+1}(X)H_{2n'+1}(X)\\&\quad=\sqrt{\pi}\sum_{n=0}\frac{(-1)^{n+1}(2n+1)}{\Gamma(n+\frac{3}{2})}H_{2n}(X).\end{aligned} \tag{44}$$

In summary, based on the technique of integration within an ordered product (IWOP) of operators [9–11] we show that operator ordering method can lead us to derive new operator identities and new integration formulas regarding to Hermite polynomials, in which (33) seems quite useful.

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